AMERICAN UNIVERSITY OF BEIRUT
Mathematics Department-FAS

MATH 251
FINAL EXAM
SPRING 2011-2012
Closed Book, 2hours

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	20	
2	20	
3	20	
4	16	
5	24	
TOTAL	100	

1. **(20 points)** Consider the following 5×5 diagonally dominant lower Hessenberg matrix

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 4 & 1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

   (a) (10 pts) Apply the Naive Gauss reduction on the matrix A showing the status of that matrix after each elimination, i.e. each of the pivot rows and the corresponding multipliers should be identified and circled.

...

(b) (3 pts) Extract out of this process, the Upper triangular matrix U and the Unit Lower triangular matrix P.

$$U = \begin{pmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

$$L = \begin{pmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

(c) (4 pts)Check that :

- at each reduction, the multipliers reduce to <u>one value</u>
- at each reduction except the last, the modified elements reduce to <u>two values</u>, in addition to <u>the diagonal element</u> at last reduction

then compute the total number of flops needed for the LU-decomposition of the matrix A. Justify your answer.

(d) (3 pts) Deduce the total number of flops needed for the LU-decomposition of the $(n \times n)$ diagonally dominant lower Hessenberg matrix B where c is a constant and

$$B = \begin{pmatrix} c & 1 & 0 & 0 & . & . & . & 0 \\ 1 & c & 1 & 0 & . & . & . & 0 \\ 1 & 1 & c & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & 1 & c & 1 \\ 1 & 1 & 1 & 1 & . & . & 1 & c \end{pmatrix}$$

Express your answer in terms of n.

2. **(20 points)** Consider the following Initial value Problem:

$$(IVP)\begin{cases} \frac{dy}{dt} = y - t^2 + 1\,; \;\; t \in [0.0, 2.0] \\ y(0) = 0.5 \end{cases}$$

To solve (IVP) in $\underline{F(10, 4, -20, +20) \text{ (rounding to the closest)}}$:

(a) (5 pts)(IVP) is first solved on [0.00 , 0.8] using **2** steps of the **discrete scheme** of the 2^{nd} order Runge Kutta method - Midpoint Rule: (RK2.M).

- Write the formulae of this scheme:

$$(RK2.M)\begin{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{i+1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

- Use **2 steps ONLY** of this scheme to approximate $y(0.4)$ and $y(0.8)$.

i	t_i	y_i	k_1	k_2	y_{i+1}
0	0.0	.	.	.	.
1	.	.	.	.	.

(b) (6 pts) (IVP) is then solved on [0.8 , 2.0] using **3** steps of the **discrete scheme** of the 2nd order Runge Kutta method - Trapezoid Rule (Heun's method): (RK2.T).

- Write the formulae of this scheme:

$$(RK2.T)\begin{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{i+1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

- Use **3 steps ONLY** of this scheme to approximate $y(1.2)$, $y(1.6)$ and $y(2.0)$.

i	t_i	y_i	k_1	k_2	y_{i+1}
0	0.8	.	.	.	.
1	.	.	.	.	.
2	.	.	.	.	.

(c) (1pt) The discrete solution of (IVP) is

$Y_5 = \{y_0 = 0.5,$ ....$\}$

(d) (8 pts)Consider the following general Initial value problem:

$$(IVP)\begin{cases} y'(t) = f(t, y(t)) & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases}$$

where the function $f(t, y(t))$ is at least a C^4 function over its domain of definition.

The 4^{th} order explicit Runge Kutta discrete scheme solving (IVP) is as follows:

$$(RK4)\begin{cases} k_1 = f(t_i, y_i) \\ k_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1) \\ k_3 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2) \\ k_4 = f(t_i + h, y_i + hk_3) \\ y_{i+1} = y_i + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

Write the MATLAB function:

$$\textbf{function y} = \mathbf{RK4(f, y_0, t_0, T, n)}$$

that inputs a function $f(t, y(t))$, the initial condition y_0 of (IVP), and a positive integer n being the number of subintervals of equal length subdividing the time interval $[t_0, T]$. This function outputs the discrete solution of (IVP) using the (RK4) discrete scheme as a vector y of length $(n+1)$.

function y $= \mathbf{RK4}(\mathbf{f}, \mathbf{y_0}, \mathbf{t_0}, \mathbf{T}, \mathbf{n})$
Compute the time step h

3. **(20 points)** Consider the following set of data:

$$D_n = \{(x_i, y_i) | i = 0, ..., n \text{ where } y_i = f(x_i) \text{ , and } x_{i+1} - x_i = h, \ 0 < h \leq 1\}$$

(a) (6 points) **Write** first the **Forward difference** formula $\phi_h(f(x_i))$ that approximates the first derivative $f'(x_i)$, then **derive** the expression of the infinite error series

$$f^{'}(x_i) - \phi_h(f(x_i)) = \epsilon(h)$$

in the form:

$$\epsilon(h) = c_1 h^{\alpha_1} + c_2 h^{\alpha_2} + c_3 h^{\alpha_3} + ...,$$

by determining the values of the constants $\{\alpha_1, \ \alpha_2, \ \alpha_3, ...\}$.

- $\phi_h(f(x_i)) =$
- $\epsilon(h) =$

(b) (6 points) Based on the Forward difference formula, **derive** Richardson extrapolation operators of orders 1 and 2 and the order of their error series.

- 1st order Richardson extrapolation operator:

  $\phi_h^1(f(x_i)) =$

  Corresponding Error = O(...........)

- 2nd order Richardson extrapolation operator:

  $\phi_h^2(f(x_i)) =$

  Corresponding Error = O(...........)

(c) (8 pts) Let $f(x) = x + e^x$. For the purpose of approximating $f'(1.0)$ and then improving this approximation, fill in the empty slots in the following table adequately, starting with $h_0 = 0.4$. Express all the results obtained in $F(10, 5, -15, +15)$.

h	$\phi_h(.)$	$\phi_h^1(.)$	$\phi_h^2(.)$
$h_0 = 0.4$			
$h_0/2$			
$h_0/4$			

Best approximation to $f'(1.0)$:

4. **(16 points)**In order to approximate the double integral

$$I = \int_{1.4}^{2} \int_{1}^{1.5} ln(x+2y)\, dy\, dx$$

over the rectangle $R = \{1 \leq y \leq 1.5\,;\, 1.4 \leq x \leq 2\}$, write first I as :

$$I = \int_{1.4}^{2} [\int_{1}^{1.5} ln(x+2y)\, dy\,]\ dx = \int_{1.4}^{2} J\, dx$$

where $J = \int_{1}^{1.5} ln(x+2y)\, dy$

- In a first step, treating **x as a constant**, approximate J using the **Composite Trapezoidal Rule**, by partitioning the interval $[1, 1.5]$ into to 2 subintervals of equal length. (Your answer should be expressed in terms of some function of x).
  $J = \int_{1}^{1.5} ln(x+2y)\, \mathbf{dy} \approx T(0.25) = g(x) =$

- Secondly, approximate $I = \int_{1.4}^{2} J\, \mathbf{dx} \approx \int_{1.4}^{2} \mathbf{g(x)}\, \mathbf{dx}$ using the **Composite Midpoint Rule**, by partitioning the interval $[1.4, 2]$ into to 2 subintervals of equal length. (Express your answer with a precision $p = 4$ rounding to the closest).
  $I = \int_{1.4}^{2} J\, dx \approx M(0.3) =$

5. **(24 points)** Consider the function $f(x) = x^2 - a$ where $a \geq 1$.

(a) (7 pts)Plot the graph of f, then write Newton's formula: $r_{n+1} = g(r_n)$ that computes the **negative root** of the function f. Is there any restriction on the initial condition r_0? Justify your answer.

(b) (7 pts)Use part (a) to compute $-\sqrt{\mathbf{3}}$ up to 5 decimal figures rounding to the closest. (Locate the root first).

(c) (5 pts) Use the general case where $f(x) = x^2 - a$, and the equation **derived in (a)** to prove that:

$$\forall n \geq 0\ ,\ (r_{n+1} - r) < 0$$

(i.e. all the elements of the sequence $\{r_n\}_{n\geq 1}$ generated by Newton's method lie to the left of r)
(Hint: Let $a = r^2$)

(d) (5 pts)Show that Newton's iterative sequence $\{r_n\}_{n\geq 1}$ **derived in (a)** is an increasing sequence converging to r. For that purpose:

- Prove that $\forall n \geq 1\ ,\ r_{n+1} - r_n > 0$

- Prove next that the sequence $\{r_n\}$ converges to r, i.e. that:

$$\lim_{n\to\infty} r_n = r$$

<u>Hint:</u> Use (c) and show first by induction on n that :

$$\frac{r_1 - r}{2^n} < ... < \frac{r_{n-1} - r}{2^2} < \frac{r_n - r}{2^1} < r_{n+1} - r < 0$$